

On the quark-mass dependence of the baryon ground-state masses

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Abstract

We perform a chiral extrapolation of the baryon octet and decuplet masses in a relativistic formulation of chiral perturbation theory. A partial summation is assumed as implied by the use of physical baryon and meson masses in the one-loop diagrams. Upon a chiral expansion, our results are consistent with strict chiral perturbation theory at the next-to-next-to-next-to-leading order. All counter terms are correlated by a large- N_c operator analysis. Our results are confronted with recent results of unquenched three-flavor lattice simulations. We adjust the parameter set to the pion-mass dependence of the nucleon and omega masses as computed by the BMW Collaboration and predict the pion-mass dependence of the remaining baryon octet and decuplet states. The current lattice simulations can be described accurately and smoothly up to pion masses of about 600 MeV. In particular, we recover the recent results of HSC without any further adjustments.

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I. INTRODUCTION

QCD lattice simulations offer the unique opportunity to determine low-energy parameters of the chiral Lagrangian. Currently, various lattice groups work on the baryon octet and decuplet masses in unquenched simulations with three light quarks [1–6]. Since the simulations are performed also at quark masses larger than those needed to reproduce the physical hadron masses, new information is generated that may be used to determine so-far unknown low-energy constants.

The strict chiral expansion of the baryon masses based on the flavor $SU(3)$ heavy-baryon formulation is poorly convergent [7–14]. This is in contrast to the two-flavor formulation, which appears to be sufficiently well converging as to justify its direct application to lattice data [15, 16]. Thus, an extrapolation of the recent lattice simulation [1–5] down to the physical limit is not straightforward and requires detailed studies. Indeed, at next-to-next-to-leading order (NNLO) the heavy-baryon formulation cannot describe the latest LHP [2] and PACS-CS [3, 17] lattice data.

A phenomenological remedy of the three-flavor convergence problem was suggested long ago by Donoghue and Holstein [8, 9, 12, 18]. It was shown that the introduction of an ultraviolet cutoff of the order of the kaon mass tames the large contribution of the one-loop effects. For a recent application of such a scheme to the chiral extrapolation of the baryon masses, we refer to Ref. [19]. In our previous work [11], we suggested a possible alternative. A partial summation scheme was suggested where physical meson and baryon masses are used in the one-loop expression constituting the NNLO effects. This leads to very reasonable results for physical quark masses. In a follow-up work [20], such a scheme was applied to a chiral extrapolation of the baryon octet and decuplet masses. Though the overall quark-mass dependence of the early lattice results of MILC [1] was recovered roughly, the self-consistency of this approach leads to a striking and unexpected phenomenon. The system of eight nonlinear coupled equations implies not necessarily a continuous quark-mass dependence of the baryon masses. In the more recent work [21], yet a different strategy was investigated. It was shown that with a phenomenological adjustment of the meson-baryon coupling constants in the one-loop baryon octet and decuplet self-energies evaluated at the NNLO level, one may get close to the simulation results of PACS-CS [3]. In a strict chiral expansion, the effect of modifying the meson-baryon coupling constants off their chiral $SU(3)$

symmetric values enters at next-to-next-to-next-to-next-to-leading order ($N^4\text{LO}$). It is yet to be clear why the $N^4\text{LO}$ effects are possibly more important than the next-to-next-to-next-to-leading order effects.

It is the purpose of the present study to extend our previous work [20] and confront it with recent unquenched lattice simulations. We will focus on the results of the BMW group [5] since they provide results that can be used directly in the continuum with negligible lattice effects. For a recent detailed study of lattice volume effects of the results of the NPLQCD Collaboration [22], we refer to Ref. [23]. The incorporation of the next-to-next-to-next-to-leading order effects in our approach will allow for a quantitative extrapolation of lattice results. At this order, the meson-baryon coupling constants are $SU(3)$ flavor symmetric. Altogether, there are 34 additional counter terms to be considered when going from the NNLO to the $N^3\text{LO}$ level. While the relevant counter terms were constructed previously in Ref. [7] for the baryon octet, analogous terms for the baryon decuplet will be presented fully for the first time in this work. Some terms relevant in the decuplet sector can be found in Refs. [15, 24–27]. Because of the large number of unknown parameters an analysis at $N^3\text{LO}$ appears futile at first. However, we will demonstrate that using large- N_c correlations of the counter terms [26–29], a meaningful extrapolation can be performed. The number of additional unknown parameters is reduced significantly.

At subleading order in the large- N_c expansion, we find the relevance of altogether 20 parameters, where we find a subset of 15 parameters to be most important. The latter are adjusted to reproduce the empirical baryon octet and decuplet masses together with the pion-mass dependence of the nucleon and omega mass as predicted by recent QCD lattice simulations of the BMW group [5]. The smooth pion-mass dependence can be reproduced accurately. A prediction for the pion-mass dependence of the remaining octet and decuplet masses is presented and confronted with available unquenched three-flavor simulations of various lattice groups.

II. CHIRAL LAGRANGIAN WITH BARYON OCTET AND DECUPLET FIELDS

The construction rules for the chiral $SU(3)$ Lagrangian density are explained in Refs. [7, 13, 30–34]. We recall the terms relevant for our work using the notation and conventions

of Refs. [11, 27]. The leading order chiral Lagrangian is of the order Q and is given by

$$\begin{aligned}\mathcal{L}^{(1)} = & \text{tr} \left\{ \bar{B} (i \not{D} - M_{[8]}) B \right\} + F \text{tr} \left\{ \bar{B} \gamma^\mu \gamma_5 [i U_\mu, B] \right\} + D \text{tr} \left\{ \bar{B} \gamma^\mu \gamma_5 \{i U_\mu, B\} \right\} \\ & - \text{tr} \left\{ \bar{B}_\mu \cdot ((i \not{D} - M_{[10]}) g^{\mu\nu} - i (\gamma^\mu D^\nu + \gamma^\nu D^\mu) + \gamma^\mu (i \not{D} + M_{[10]}) \gamma^\nu) B_\nu \right\} \\ & + C \left(\text{tr} \left\{ (\bar{B}_\mu \cdot i U^\mu) B \right\} + \text{H.c.} \right) + H \text{tr} \left\{ (\bar{B}^\mu \cdot \gamma_\nu \gamma_5 B_\mu) i U^\nu \right\},\end{aligned}\quad (1)$$

with $U_\mu = i \partial_\mu \Phi / (2f) + \dots$. The leading order baryon masses are given by $M_{[8]}$ and $M_{[10]}$ for the members of the flavor $SU(3)$ octet and decuplet, respectively. The parameters F and D in Eq. (1) may be determined from the study of semileptonic decays of baryons, $B \rightarrow B' + e + \bar{\nu}_e$. This leads to $F \simeq 0.45$ and $D \simeq 0.80$ [35, 36], the values used in this work. The value of C may be extracted from the hadronic decays of the decuplet baryons. We recall from Ref. [36] the empirical value $C = 1.6$. The parameter H is poorly determined by experimental data so far [37]. Using large- N_c sum rules, the parameters C and H may be also estimated given the empirical values for F and D [38]. It holds

$$H = 9F - 3D, \quad C = 2D, \quad (2)$$

at subleading order in the $1/N_c$ expansion.

A complete list of chiral symmetry-conserving Q^2 counter terms, relevant for the calculation of the N³LO baryon mass corrections, was given in Refs. [27, 36]. Following these works, we group the Q^2 counter terms according to their Dirac structure and display the relevant terms only. It holds:

$$\mathcal{L}^{(2)} = \mathcal{L}^{(S)} + \mathcal{L}^{(V)}, \quad (3)$$

with

$$\begin{aligned}\mathcal{L}^{(S)} = & -\frac{1}{2} g_0^{(S)} \text{tr} \left\{ \bar{B} B \right\} \text{tr} \left\{ U_\mu U^\mu \right\} - \frac{1}{2} g_1^{(S)} \text{tr} \left\{ \bar{B} U^\mu \right\} \text{tr} \left\{ U_\mu B \right\} \\ & - \frac{1}{4} g_D^{(S)} \text{tr} \left\{ \bar{B} \{ \{ U_\mu, U^\mu \}, B \} \right\} - \frac{1}{4} g_F^{(S)} \text{tr} \left\{ \bar{B} [\{ U_\mu, U^\mu \}, B] \right\} \\ & + \frac{1}{2} h_1^{(S)} \text{tr} \left\{ \bar{B}_\mu \cdot B^\mu \right\} \text{tr} \left\{ U_\nu U^\nu \right\} + \frac{1}{2} h_2^{(S)} \text{tr} \left\{ \bar{B}_\mu \cdot B^\nu \right\} \text{tr} \left\{ U^\mu U_\nu \right\} \\ & + h_3^{(S)} \text{tr} \left\{ \left(\bar{B}_\mu \cdot B^\mu \right) \left(U^\nu U_\nu \right) \right\} + \frac{1}{2} h_4^{(S)} \text{tr} \left\{ \left(\bar{B}_\mu \cdot B^\nu \right) \{ U^\mu, U_\nu \} \right\} \\ & + h_5^{(S)} \text{tr} \left\{ \left(\bar{B}_\mu \cdot U_\nu \right) \left(U^\nu \cdot B^\mu \right) \right\} \\ & + \frac{1}{2} h_6^{(S)} \text{tr} \left\{ \left(\bar{B}_\mu \cdot U^\mu \right) \left(U^\nu \cdot B_\nu \right) + \left(\bar{B}_\mu \cdot U^\nu \right) \left(U^\mu \cdot B_\nu \right) \right\},\end{aligned}$$

$$\begin{aligned}
\mathcal{L}^{(V)} = & -\frac{1}{4} g_0^{(V)} \left(\text{tr} \left\{ \bar{B} i \gamma^\mu D^\nu B \right\} \text{tr} \left\{ U_\nu U_\mu \right\} + \text{H.c.} \right) \\
& -\frac{1}{8} g_1^{(V)} \left(\text{tr} \left\{ \bar{B} U_\mu \right\} i \gamma^\mu \text{tr} \left\{ U_\nu D^\nu B \right\} + \text{tr} \left\{ \bar{B} U_\nu \right\} i \gamma^\mu \text{tr} \left\{ U_\mu D^\nu B \right\} + \text{H.c.} \right) \\
& -\frac{1}{8} g_D^{(V)} \left(\text{tr} \left\{ \bar{B} i \gamma^\mu \left\{ \{U_\mu, U_\nu\}, D^\nu B \right\} \right\} + \text{H.c.} \right) \\
& -\frac{1}{8} g_F^{(V)} \left(\text{tr} \left\{ \bar{B} i \gamma^\mu \left[\{U_\mu, U_\nu\}, D^\nu B \right] \right\} + \text{H.c.} \right) \\
& +\frac{1}{4} h_1^{(V)} \left(\text{tr} \left\{ \bar{B}_\lambda \cdot i \gamma^\mu D^\nu B^\lambda \right\} \text{tr} \left\{ U_\mu U_\nu \right\} + \text{H.c.} \right) \\
& +\frac{1}{4} h_2^{(V)} \left(\text{tr} \left\{ (\bar{B}_\lambda \cdot i \gamma^\mu D^\nu B^\lambda) \{U_\mu, U_\nu\} \right\} + \text{H.c.} \right) \\
& +\frac{1}{4} h_3^{(V)} \left(\text{tr} \left\{ (\bar{B}_\lambda \cdot U_\mu) i \gamma^\mu (U_\nu \cdot D^\nu B^\lambda) + (\bar{B}_\lambda \cdot U_\nu) i \gamma^\mu (U_\mu \cdot D^\nu B^\lambda) \right\} + \text{H.c.} \right). \quad (4)
\end{aligned}$$

The large number of unknown chiral parameters at this order is reduced by matching the low-energy and the $1/N_c$ expansions of the product of two axial-vector quark currents [27, 36].

The 17 parameters in Eq. (4) are correlated by the 12 sum rules

$$\begin{aligned}
g_F^{(S)} &= g_0^{(S)} - \frac{1}{2} g_1^{(S)}, \quad h_1^{(S)} = 0, \quad h_2^{(S)} = 0, \quad h_3^{(S)} = \frac{3}{2} g_0^{(S)} - \frac{9}{4} g_1^{(S)} + \frac{1}{2} g_D^{(S)}, \\
h_4^{(S)} &= 3 \left(g_D^{(S)} + \frac{3}{2} g_1^{(S)} \right), \quad h_5^{(S)} = g_D^{(S)} + 3 g_1^{(S)}, \quad h_6^{(S)} = -3 \left(g_D^{(S)} + \frac{3}{2} g_1^{(S)} \right), \\
g_D^{(V)} &= -\frac{3}{2} g_1^{(V)}, \quad g_F^{(V)} = g_0^{(V)} - \frac{1}{2} g_1^{(V)}, \quad h_1^{(V)} = 0, \quad h_2^{(V)} = \frac{3}{2} g_0^{(V)} - 3 g_1^{(V)}, \\
h_3^{(V)} &= \frac{3}{2} g_1^{(V)}, \quad (5)
\end{aligned}$$

leaving only the five unknown parameters, $g_0^{(S)}, g_1^{(S)}, g_D^{(S)}$ and $g_0^{(V)}, g_1^{(V)}$.

It remains to detail the explicit symmetry-breaking terms. We collect the terms relevant for the computation of the baryon self-energies at N³LO. There are five Q^2 terms:

$$\begin{aligned}
\mathcal{L}_\chi^{(2)} &= 2 b_0 \text{tr} (\bar{B} B) \text{tr} (\chi_+) + 2 b_D \text{tr} (\bar{B} \{ \chi_+, B \}) + 2 b_F \text{tr} (\bar{B} [\chi_+, B]) \\
&\quad - 2 d_0 \text{tr} (\bar{B}_\mu \cdot B^\mu) \text{tr} (\chi_+) - 2 d_D \text{tr} ((\bar{B}_\mu \cdot B^\mu) \chi_+), \\
\chi_+ &= \chi_0 - \frac{1}{8 f^2} \{ \Phi, \{ \Phi, \chi_0 \} + \mathcal{O}(\Phi^4) \}, \quad (6)
\end{aligned}$$

with $\chi_0 = 2 B_0 \text{diag}(m, m, m_s)$ proportional to the quark-mass matrix. We do not consider isospin-violating effects in this work, and we use $f = 92.4$ MeV. A matching of the chiral interaction terms (6) to the large- N_c operator analysis for the baryon masses in Refs. [26, 28] leads to one sum rule:

$$b_D + b_F = \frac{1}{3} d_D, \quad (7)$$

accurate to subleading order in the $1/N_c$ expansion.

We continue with the symmetry-breaking part of the chiral Lagrangian. There are five Q^3 terms and 12 Q^4 terms:

$$\begin{aligned}
\mathcal{L}_\chi^{(3)} &= \zeta_0 \text{tr}(\bar{B}(i\not{D} - M_{[8]})B) \text{tr}(\chi_+) + \zeta_D \text{tr}(\bar{B}(i\not{D} - M_{[8]})[B, \chi_+]) \\
&+ \zeta_F \text{tr}(\bar{B}(i\not{D} - M_{[8]})\{B, \chi_+\}) \\
&- \xi_0 \text{tr}(\bar{B}_\mu(i\not{D} - M_{[10]})B^\mu) \text{tr}(\chi_+) - \xi_D \text{tr}(\bar{B}_\mu(i\not{D} - M_{[10]})B^\mu \chi_+) , \\
\mathcal{L}_\chi^{(4)} &= c_0 \text{tr}(\bar{B}B) \text{tr}(\chi_+^2) + c_1 \text{tr}(\bar{B}\chi_+) \text{tr}(\chi_+ B) \\
&+ c_2 \text{tr}(\bar{B}\{\chi_+^2, B\}) + c_3 \text{tr}(\bar{B}[\chi_+^2, B]) \\
&+ c_4 \text{tr}(\bar{B}\{\chi_+, B\}) \text{tr}(\chi_+) + c_5 \text{tr}(\bar{B}[\chi_+, B]) \text{tr}(\chi_+) \\
&+ c_6 \text{tr}(\bar{B}B) (\text{tr}(\chi_+))^2 \\
&- e_0 \text{tr}(\bar{B}_\mu \cdot B^\mu) \text{tr}(\chi_+^2) - e_1 \text{tr}((\bar{B}_\mu \cdot \chi_+)(\chi_+ \cdot B^\mu)) \\
&- e_2 \text{tr}((\bar{B}_\mu \cdot B^\mu) \cdot \chi_+^2) - e_3 \text{tr}((\bar{B}_\mu \cdot B^\mu) \cdot \chi_+) \text{tr}(\chi_+) \\
&- e_4 \text{tr}(\bar{B}_\mu \cdot B^\mu) (\text{tr}(\chi_+))^2 .
\end{aligned} \tag{8}$$

We consider again large- N_c sum rules for the parameters introduced in Eq. (8). For $\zeta_0, \zeta_D, \zeta_F$, and ξ_0, ξ_D , the sum rules are analogous to the ones for the Q^2 terms, i.e. it holds:

$$\zeta_D + \zeta_F = \frac{1}{3} \xi_D , \tag{9}$$

at subleading order. A matching of the chiral interaction terms (8) to the large- N_c operator analysis for the baryon masses in Ref. [26] leads to the seven sum rules:

$$\begin{aligned}
c_0 &= \frac{1}{2} c_1, & c_2 &= -\frac{3}{2} c_1, & c_3 &= 0, \\
e_0 &= 0, & e_1 &= -2 c_2, & e_2 &= 3 c_2, & e_3 &= 3(c_4 + c_5),
\end{aligned} \tag{10}$$

valid at NNLO in the expansion. Assuming the approximate validity of Eq. (10), it suffices to determine the five parameters c_1, c_4, c_5, c_6 , and e_4 .

The terms in $\mathcal{L}_\chi^{(3)}$ are redundant upon a suitable redefinition of the baryon fields. Why do we consider such terms at all? The reason is that the redundancy of the parameters $\zeta_0, \zeta_D, \zeta_F$, and ξ_0, ξ_D is lifted once we insist on the large- N_c relations (10). This is seen by eliminating $\mathcal{L}_\chi^{(3)}$ in application of the equation of motion for the baryon fields. A renormalization of terms already present in the chiral Lagrangian, in particular, those in $\mathcal{L}_\chi^{(4)}$, arises. We find

the renormalized or effective coupling strengths:

$$\begin{aligned}
c_0^{\text{eff}} &= -2(\zeta_D b_D - \zeta_F b_F) + c_0, & e_0^{\text{eff}} &= e_0, \\
c_1^{\text{eff}} &= -\frac{4}{3}(\zeta_D b_D - 3\zeta_F b_F) + c_1, & e_1^{\text{eff}} &= \frac{4}{3}\xi_D d_D + e_1, \\
c_2^{\text{eff}} &= 2(\zeta_D b_D - 3\zeta_F b_F) + c_2, & e_2^{\text{eff}} &= -2\xi_D d_D + e_2, \\
c_3^{\text{eff}} &= -2(\zeta_D b_F + \zeta_F b_D) + c_3, & e_3^{\text{eff}} &= -2(\xi_0 d_D + \xi_D d_0) + e_3, \\
c_4^{\text{eff}} &= -2(\zeta_0 b_D + \zeta_D b_0 + 2\zeta_D b_D - 2\zeta_F b_F) + c_4, & e_4^{\text{eff}} &= -2\xi_0 d_0 + e_4, \\
c_5^{\text{eff}} &= -2(\zeta_0 b_F + \zeta_F b_0) + c_5, \\
c_6^{\text{eff}} &= -2(\zeta_0 b_0 - \zeta_D b_D + \zeta_F b_F) + c_6.
\end{aligned} \tag{11}$$

Inserting Eq. (11) into the sum rules (10) reveals that there is no nontrivial way to dial the parameters ζ_D, ζ_F , and ξ_D as to be compatible with Eq. (10). It follows that those parameters are independent of the five parameters c_1, c_4, c_5, c_6 and e_4 . For the singlet parameters one finds the correlation

$$3\zeta_0(b_D + b_F) = \xi_0 d_D, \tag{12}$$

which is consistent with the trivial solution $\zeta_0 = 0 = \xi_0$. The results (11, 12) illustrate that the N³LO effect of a variation of the parameters ζ_0 and ξ_0 as correlated by Eq. (12) can be reproduced by a suitable variation of the parameters $c_{4,5,6}$ and e_4 . We conclude that given the sum rules, (9, 10) there are altogether 8 symmetry-breaking parameters at N³LO: the parameters c_1, c_4, c_5, c_6, e_4 , and ζ_F, ξ_D together with either ξ_0 or ζ_0 .

In anticipation of our results, we state the crucial importance of the parameters ζ_F, ξ_D . Within the self-consistent approach applied in this work, they lead to a smooth chiral extrapolation of the baryon masses. Any attempt with $\zeta_F = \xi_D = 0$ to establish a smooth chiral extrapolation would be futile.

III. CHIRAL LOOP EXPANSION OF THE BARYON MASSES

We turn to the computation of the baryon masses. The baryon self-energy, $\Sigma_B(\not{p})$, may be considered to be a function of $p_\mu \gamma^\mu$ only, with the 4-momentum p_μ of the baryon B . This is obvious for the spin one-half baryons, but less immediate for the spin-three-half baryons. We refer to Ref. [11] for technical details. To order Q^4 , the self-energy receives contributions

from tree-level diagrams and one-loop diagrams:

$$\Sigma_B(M_B) = \Sigma_B^{\text{tree-level}} + \Sigma_B^{\text{loop}}, \quad (13)$$

where the index B stands for the members of the flavor $SU(3)$ octet and decuplet, $B \in [8], [10]$. The separation of the baryon self-energies into a loop and a tree-level contribution is not unique depending on the renormalization scheme. In this work, we apply the χ MS scheme developed in Ref. [11], where we keep the dependence on the ultraviolet renormalization scale only. A possible dependence on an infrared renormalization scale is ignored in this work as to be close to more conventional renormalization schemes. A matching with alternative renormalization schemes is most economically performed by a direct comparison with the explicit expressions of this section. Our renormalized tree-level self-energies are collected in the Appendix.

The physical mass of the baryon M_B is determined by the condition

$$M_B - \Sigma_B(M_B) = \begin{cases} \bar{M}_{[8]} & \text{for } B \in [8] \\ \bar{M}_{[10]} & \text{for } B \in [10] \end{cases}, \quad (14)$$

where $\bar{M}_{[8]}$ and $\bar{M}_{[10]}$ are the renormalized and scale-independent bare masses of the baryon octet and decuplet. We consistently use a bar for renormalized quantities throughout this work. At NNLO, the one-loop contributions probe the coupling constants F, D, C , and H introduced in Eq. (1). Complete expressions for baryon octet and decuplet states were first established in Ref. [11]. Partial results are documented in Ref. [39]. Here, we complement our previous result by additional contributions from the counter terms (4, 8), which turn relevant at N³LO. For previous N³LO studies in the baryon octet sector see Refs. [7, 40]. Some partial results in the baryon decuplet sector can be found in [41]. Altogether, we obtain for the renormalized loop contribution the expressions

$$\begin{aligned} \Sigma_{B \in [8]}^{\text{loop}} = & \sum_{Q \in [8], R \in [8]} \left(\frac{G_{QR}^{(B)}}{2f} \right)^2 \left\{ - \frac{(M_B + M_R)^2}{E_R + M_R} p_{QR}^2 \left(\bar{I}_{QR} + \frac{\bar{I}_Q}{M_R^2 - m_Q^2} \right) + \frac{M_R^2 - M_B^2}{2 M_B} \bar{I}_Q \right\} \\ & + \sum_{Q \in [8], R \in [10]} \left(\frac{G_{QR}^{(B)}}{2f} \right)^2 \left\{ - \frac{2}{3} \frac{M_B^2}{M_R^2} (E_R + M_R) p_{QR}^2 \left(\bar{I}_{QR} + \frac{\bar{I}_Q}{M_R^2 - m_Q^2} \right) \right. \\ & \quad \left. + \left(\frac{(M_R - M_B)(M_R + M_B)^3 + m_Q^4}{12 M_B M_R^2} + \frac{5 M_B^2 + 6 M_R M_B - 2 M_R^2}{12 M_B M_R^2} m_Q^2 \right) \bar{I}_Q \right\} \\ & + \frac{1}{(2f)^2} \sum_{Q \in [8]} \left(G_{BQ}^{(\chi)} - m_Q^2 G_{BQ}^{(S)} - \frac{1}{4} m_Q^2 M_B G_{BQ}^{(V)} \right) \bar{I}_Q, \end{aligned} \quad (15)$$

and

$$\begin{aligned}
\Sigma_{B \in [10]}^{\text{loop}} = & \sum_{Q \in [8], R \in [8]} \left(\frac{G_{QR}^{(B)}}{2f} \right)^2 \left\{ -\frac{1}{3} (E_R + M_R) p_{QR}^2 \left(\bar{I}_{QR} + \frac{\bar{I}_Q}{M_R^2 - m_Q^2} \right) \right. \\
& + \left(\frac{(M_R - M_B)(M_R + M_B)^3 + m_Q^4}{24 M_B^3} - \frac{3 M_B^2 + 2 M_R M_B + 2 M_R^2}{24 M_B^3} m_Q^2 \right) \bar{I}_Q \Big\} \\
& + \sum_{Q \in [8], R \in [10]} \left(\frac{G_{QR}^{(B)}}{2f} \right)^2 \left\{ -\frac{(M_B + M_R)^2}{9 M_R^2} \frac{2 E_R (E_R - M_R) + 5 M_R^2}{E_R + M_R} p_{QR}^2 \left(\bar{I}_{QR} (M_B^2) \right. \right. \\
& + \frac{\bar{I}_Q}{M_R^2 - m_Q^2} \Big) + \left(\frac{M_R^4 + M_B^4 + 12 M_R^2 M_B^2 - 2 M_R M_B (M_B^2 + M_R^2)}{36 M_B^3 M_R^2} (M_R^2 - M_B^2) \right. \\
& + \frac{(M_B + M_R)^2 m_Q^4}{36 M_B^3 M_R^2} + \frac{3 M_B^4 - 2 M_B^3 M_R + 3 M_B^2 M_R^2 - 2 M_R^4}{36 M_B^3 M_R^2} m_Q^2 \Big) \bar{I}_Q \Big\} \\
& + \frac{1}{(2f)^2} \sum_{Q \in [8]} \left(G_{BQ}^{(\chi)} - m_Q^2 G_{BQ}^{(S)} - \frac{1}{4} m_Q^2 M_B G_{BQ}^{(V)} \right) \bar{I}_Q, \tag{16}
\end{aligned}$$

where

$$\begin{aligned}
\bar{I}_Q &= \frac{m_Q^2}{(4\pi)^2} \ln \left(\frac{m_Q^2}{\mu_{UV}^2} \right), \\
\bar{I}_{QR} &= \frac{1}{16\pi^2} \left\{ \left(\frac{1}{2} \frac{m_Q^2 + M_R^2}{m_Q^2 - M_R^2} - \frac{m_Q^2 - M_R^2}{2 M_B^2} \right) \ln \left(\frac{m_Q^2}{M_R^2} \right) \right. \\
&+ \frac{p_{QR}}{M_B} \left(\ln \left(1 - \frac{M_B^2 - 2 p_{QR} M_B}{m_Q^2 + M_R^2} \right) - \ln \left(1 - \frac{M_B^2 + 2 p_{QR} M_B}{m_Q^2 + M_R^2} \right) \right) \Big\}, \\
p_{QR}^2 &= \frac{M_B^2}{4} - \frac{M_R^2 + m_Q^2}{2} + \frac{(M_R^2 - m_Q^2)^2}{4 M_B^2}, \quad E_R^2 = M_R^2 + p_{QR}^2. \tag{17}
\end{aligned}$$

The sums in Eqs. (15, 16) extend over the intermediate Goldstone bosons ($Q \in [8]$) baryon octet ($R \in [8]$) and decuplet states ($R \in [10]$). The coupling constants $G_{QR}^{(B)}$ are determined by the parameters F, D, C, H . They are listed in Ref. [11]. The coupling constants $G_{QR}^{(\chi)}$ probe the renormalized symmetry-breaking parameters $\bar{b}_0, \bar{b}_D, \bar{b}_F, \bar{d}_0$, and \bar{d}_D . They are detailed in Table I, together with $G_{QR}^{(S)}$ and $G_{QR}^{(V)}$ which are proportional to the symmetry-preserving parameters introduced in Eq. (4). In Table I, we apply the notation

$$\tilde{h}_1^{(S)} \equiv h_1^{(S)} + \frac{1}{4} h_2^{(S)}, \quad \tilde{h}_2^{(S)} \equiv h_3^{(S)} + \frac{1}{4} h_4^{(S)}, \quad \tilde{h}_3^{(S)} \equiv h_5^{(S)} + \frac{1}{4} h_6^{(S)}. \tag{18}$$

The 6 parameters $h_{1-6}^{(S)}$ enter the decuplet self-energy in the three combinations (18) only.

The mesonic tadpole \bar{I}_Q has a logarithmic dependence on the ultraviolet renormalization scale μ_{UV} . The ultraviolet scale dependence of Eqs. (15, 16) is counteracted by the to-be-

$B \quad Q$	$G_{BQ}^{(\chi)}$	$G_{BQ}^{(S)}$	$G_{BQ}^{(V)}$
π	$24 B_0 m (2 \bar{b}_0 + \bar{b}_D + \bar{b}_F)$	$3 g_0^{(S)} + \frac{3}{2} g_D^{(S)} + \frac{3}{2} g_F^{(S)}$	$3 g_0^{(V)} + \frac{3}{2} g_D^{(V)} + \frac{3}{2} g_F^{(V)}$
$N \quad K$	$8 B_0 (m + m_s) (4 \bar{b}_0 + 3 \bar{b}_D - \bar{b}_F)$	$4 g_0^{(S)} + g_1^{(S)} + 3 g_D^{(S)} - g_F^{(S)}$	$4 g_0^{(V)} + g_1^{(V)} + 3 g_D^{(V)} - g_F^{(V)}$
η	$\frac{8}{3} B_0 m (2 \bar{b}_0 + \bar{b}_D + \bar{b}_F)$ $+ \frac{32}{3} B_0 m_s (\bar{b}_0 + \bar{b}_D - \bar{b}_F)$	$g_0^{(S)} + \frac{5}{6} g_D^{(S)} - \frac{1}{2} g_F^{(S)}$	$g_0^{(V)} + \frac{5}{6} g_D^{(V)} - \frac{1}{2} g_F^{(V)}$
π	$16 B_0 m (3 \bar{b}_0 + \bar{b}_D)$	$3 g_0^{(S)} + g_D^{(S)}$	$3 g_0^{(V)} + g_D^{(V)}$
$\Lambda \quad K$	$\frac{16}{3} B_0 (m + m_s) (6 \bar{b}_0 + 5 \bar{b}_D)$	$4 g_0^{(S)} + \frac{10}{3} g_D^{(S)}$	$4 g_0^{(V)} + \frac{10}{3} g_D^{(V)}$
η	$\frac{16}{9} B_0 m (3 \bar{b}_0 + \bar{b}_D)$ $+ \frac{32}{9} B_0 m_s (3 \bar{b}_0 + 4 \bar{b}_D)$	$g_0^{(S)} + g_1^{(S)} + g_D^{(S)}$	$g_0^{(V)} + g_1^{(V)} + g_D^{(V)}$
π	$48 B_0 m (\bar{b}_0 + \bar{b}_D)$	$3 g_0^{(S)} + g_1^{(S)} + 3 g_D^{(S)}$	$3 g_0^{(V)} + g_1^{(V)} + 3 g_D^{(V)}$
$\Sigma \quad K$	$16 B_0 (m + m_s) (2 \bar{b}_0 + \bar{b}_D)$	$4 g_0^{(S)} + 2 g_D^{(S)}$	$4 g_0^{(V)} + 2 g_D^{(V)}$
η	$\frac{16}{3} B_0 (m (\bar{b}_0 + \bar{b}_D) + 2 m_s \bar{b}_0)$	$g_0^{(S)} + \frac{1}{3} g_D^{(S)}$	$g_0^{(V)} + \frac{1}{3} g_D^{(V)}$
π	$24 B_0 m (2 \bar{b}_0 + \bar{b}_D - \bar{b}_F)$	$3 g_0^{(S)} + \frac{3}{2} g_D^{(S)} - \frac{3}{2} g_F^{(S)}$	$3 g_0^{(V)} + \frac{3}{2} g_D^{(V)} - \frac{3}{2} g_F^{(V)}$
$\Xi \quad K$	$8 B_0 (m + m_s) (4 \bar{b}_0 + 3 \bar{b}_D + \bar{b}_F)$	$4 g_0^{(S)} + g_1^{(S)} + 3 g_D^{(S)} + g_F^{(S)}$	$4 g_0^{(V)} + g_1^{(V)} + 3 g_D^{(V)} + g_F^{(V)}$
η	$\frac{8}{3} B_0 m (2 \bar{b}_0 + \bar{b}_D - \bar{b}_F)$ $+ \frac{32}{3} B_0 m_s (\bar{b}_0 + \bar{b}_D + \bar{b}_F)$	$g_0^{(S)} + \frac{5}{6} g_D^{(S)} + \frac{1}{2} g_F^{(S)}$	$g_0^{(V)} + \frac{5}{6} g_D^{(V)} + \frac{1}{2} g_F^{(V)}$
π	$24 B_0 m (2 \bar{d}_0 + \bar{d}_D)$	$3 \tilde{h}_1^{(S)} + 3 \tilde{h}_2^{(S)} + 2 \tilde{h}_3^{(S)}$	$3 h_1^{(V)} + 3 h_2^{(V)} + 2 h_3^{(V)}$
$\Delta \quad K$	$8 B_0 (m + m_s) (4 \bar{d}_0 + \bar{d}_D)$	$4 \tilde{h}_1^{(S)} + 2 \tilde{h}_2^{(S)} + 2 \tilde{h}_3^{(S)}$	$4 h_1^{(V)} + 2 h_2^{(V)} + 2 h_3^{(V)}$
η	$\frac{8}{3} B_0 (m (2 \bar{d}_0 + \bar{d}_D) + 4 m_s \bar{d}_0)$	$\tilde{h}_1^{(S)} + \frac{1}{3} \tilde{h}_2^{(S)}$	$h_1^{(V)} + \frac{1}{3} h_2^{(V)}$
π	$16 B_0 m (3 \bar{d}_0 + \bar{d}_D)$	$3 \tilde{h}_1^{(S)} + 2 \tilde{h}_2^{(S)} + \frac{5}{3} \tilde{h}_3^{(S)}$	$3 h_1^{(V)} + 2 h_2^{(V)} + \frac{5}{3} h_3^{(V)}$
$\Sigma^* \quad K$	$\frac{32}{3} B_0 (m + m_s) (3 \bar{d}_0 + \bar{d}_D)$	$4 \tilde{h}_1^{(S)} + \frac{8}{3} \tilde{h}_2^{(S)} + \frac{4}{3} \tilde{h}_3^{(S)}$	$4 h_1^{(V)} + \frac{8}{3} h_2^{(V)} + \frac{4}{3} h_3^{(V)}$
η	$\frac{16}{9} B_0 (m + 2 m_s) (3 \bar{d}_0 + \bar{d}_D)$	$\tilde{h}_1^{(S)} + \frac{2}{3} \tilde{h}_2^{(S)} + \tilde{h}_3^{(S)}$	$h_1^{(V)} + \frac{2}{3} h_2^{(V)} + h_3^{(V)}$
π	$8 B_0 m (6 \bar{d}_0 + \bar{d}_D)$	$3 \tilde{h}_1^{(S)} + \tilde{h}_2^{(S)} + \tilde{h}_3^{(S)}$	$3 h_1^{(V)} + h_2^{(V)} + h_3^{(V)}$
$\Xi^* \quad K$	$\frac{8}{3} B_0 (m + m_s) (12 \bar{d}_0 + 5 \bar{d}_D)$	$4 \tilde{h}_1^{(S)} + \frac{10}{3} \tilde{h}_2^{(S)} + 2 \tilde{h}_3^{(S)}$	$4 h_1^{(V)} + \frac{10}{3} h_2^{(V)} + 2 h_3^{(V)}$
η	$\frac{8}{9} B_0 m (6 \bar{d}_0 + \bar{d}_D)$ $+ \frac{8}{9} B_0 m_s (12 \bar{d}_0 + 8 \bar{d}_D)$	$\tilde{h}_1^{(S)} + \tilde{h}_2^{(S)} + \tilde{h}_3^{(S)}$	$h_1^{(V)} + h_2^{(V)} + h_3^{(V)}$
π	$48 B_0 m \bar{d}_0$	$3 \tilde{h}_1^{(S)}$	$3 h_1^{(V)}$
$\Omega \quad K$	$16 B_0 (m + m_s) (2 \bar{d}_0 + \bar{d}_D)$	$4 \tilde{h}_1^{(S)} + 4 \tilde{h}_2^{(S)} + 4 \tilde{h}_3^{(S)}$	$4 h_1^{(V)} + 4 h_2^{(V)} + 4 h_3^{(V)}$
η	$\frac{16}{3} B_0 (m \bar{d}_0 + 2 m_s (\bar{d}_0 + \bar{d}_D))$	$\tilde{h}_1^{(S)} + \frac{4}{3} \tilde{h}_2^{(S)}$	$h_1^{(V)} + \frac{4}{3} h_2^{(V)}$

TABLE I: Coefficients $G_{BQ}^{(\chi)}$, $G_{BQ}^{(S)}$ and $G_{BQ}^{(V)}$.

	Empirical value at $\mu_{UV} = 0.8 \text{ GeV}$
$2 L_6 - L_4$	-0.1×10^{-3}
$2 L_8 - L_5$	$+0.4 \times 10^{-3}$
$3 L_7 + L_8$	-0.3×10^{-3}
$m B_0$	$9.91 \times 10^{-3} \text{ GeV}^2$
$m_s B_0$	$237 \times 10^{-3} \text{ GeV}^2$
f	92.4 MeV

TABLE II: Low-energy coupling constants used in this work.

specified tree-level self-energy of Eq. (13). Applying a further chiral expansion to Eq. (13), it was demonstrated that the physical masses are renormalization-scale-independent. Here, we generalized that result to order N³LO.

In the loop expressions (15, 16), we use the meson masses accurate to the NNLO as derived in Refs. [42, 43]. We recall

$$\begin{aligned}
m_\pi^2 &= \frac{2 B_0 m}{f^2} \left\{ f^2 + \frac{1}{2} \bar{I}_\pi - \frac{1}{6} \bar{I}_\eta + 16 B_0 \left[(2m + m_s)(2L_6 - L_4) + m(2L_8 - L_5) \right] \right\}, \\
m_K^2 &= \frac{B_0(m + m_s)}{f^2} \left\{ f^2 + \frac{1}{3} \bar{I}_\eta \right. \\
&\quad \left. + 16 B_0 \left[(2m + m_s)(2L_6 - L_4) + \frac{1}{2} (m + m_s)(2L_8 - L_5) \right] \right\}, \\
m_\eta^2 &= \frac{2 B_0(m + 2m_s)}{3 f^2} \left\{ f^2 + \bar{I}_K - \frac{2}{3} \bar{I}_\eta \right. \\
&\quad \left. + 16 B_0 \left[(2m + m_s)(2L_6 - L_4) + \frac{1}{3} (m + 2m_s)(2L_8 - L_5) \right] \right\} \\
&\quad + \frac{2 B_0 m}{f^2} \left[\frac{1}{6} \bar{I}_\eta - \frac{1}{2} \bar{I}_\pi + \frac{1}{3} \bar{I}_K \right] + \frac{128}{9} \frac{B_0^2 (m - m_s)^2}{f^2} (3L_7 + L_8), \tag{19}
\end{aligned}$$

with the renormalized mesonic tadpole integrals $\bar{I}_\pi, \bar{I}_K, \bar{I}_\eta$ as given in Eq. (17). The empirical values for the three relevant combinations $2 L_6 - L_4, 2 L_8 - L_5, 3 L_7 + L_8$ are given in Table II at the renormalization scale $\mu_{UV} = 800 \text{ MeV}$.

We provide a more specific discussion of the renormalized baryon mass parameters that enter the mass equation (14). With $\bar{M}_{[8]}$ and $\bar{M}_{[10]}$, we denote the renormalized form of the bare parameters $M_{[8]}$ and $M_{[10]}$. They do not coincide with the chiral SU(3) limit of the

baryon masses. The latter are determined by a set of nonlinear and coupled equations:

$$\begin{aligned}
M &= \bar{M}_{[8]} - \frac{5 C^2}{768 \pi^2 f^2} \frac{\Delta^3 (2 M + \Delta)^3}{M^2 (M + \Delta)^2} \left\{ M + \Delta \right. \\
&\quad \left. + \frac{2 M (M + \Delta) + \Delta^2}{2 M} \right\} \ln \frac{\Delta^2 (2 M + \Delta)^2}{(M + \Delta)^4}, \\
M + \Delta &= \bar{M}_{[10]} - \frac{C^2}{384 \pi^2 f^2} \frac{\Delta^3 (2 M + \Delta)^3}{(M + \Delta)^4} \left\{ M \right. \\
&\quad \left. + \frac{2 M (M + \Delta) + \Delta^2}{2 (M + \Delta)} \right\} \ln \frac{M^4}{\Delta^2 (2 M + \Delta)^2}, \tag{20}
\end{aligned}$$

where we identified the baryon octet and decuplet masses in the one-loop self-energy with M and $M + \Delta$, respectively. The baryon masses receive contributions from the renormalized tree-level terms (14) and from the renormalized one-loop self-energies (15, 16). The sum of both provides the scale-invariant chiral limit values of the octet and decuplet masses, M and $M + \Delta$. Given any values for $\bar{M}_{[8]}$ and $\bar{M}_{[10]}$, the chiral mass parameters M and Δ are obtained by a numerical solution of Eq. (20). Though in the chiral limit the self-consistency condition (14) is not a significant effect, a perturbative expansion of Eq. (20) in Δ/M is rapidly converging, this changes as we turn on flavor-breaking effects and increase the quark masses.

We emphasize again that Eqs. (15, 16) depend on the physical meson and baryon masses m_Q and M_R . This defines a self-consistent summation since the masses of the intermediate baryon states in Eqs. (15) and (16) should match the total masses. The baryon masses are a solution of a set of eight coupled and nonlinear equations in the present scheme. This is a consequence of self-consistency imposed on the partial summation approach. The latter is a crucial requirement since the loop functions depend sensitively on the precise values of the baryon masses.

We affirm that a strict chiral expansion of the expressions (15, 16, A1, A2) to N³LO leads to results that are renormalization scale-independent.

IV. QUARK-MASS DEPENDENCE OF THE BARYON MASSES

We discuss the determination of the parameter set. In this work, we introduced altogether 41 parameters. At LO there are 2 parameters $\bar{M}_{[8]}$ and $\bar{M}_{[10]}$. At next-to-leading order, there

are the 5 parameters $\bar{b}_{0,D,F}$ and $\bar{d}_{0,D}$, together with $f = 92.4$ MeV. The parameters F, D, C , and H turn relevant at NNLO. We use the large- N_c sum rules (2) together with $F = 0.45$ and $D = 0.8$. At N³LO, there are 12+5 symmetry-breaking parameters $\bar{c}_{0-6}, \bar{e}_{0-4}$ and $\bar{\zeta}_{0,D,F}, \bar{\xi}_{0,D}$ introduced in Eq. (8) and the 17 symmetry-conserving parameters of Eq. (4). As discussed in detail in Sec. II, some of the symmetry-breaking parameters are redundant at N³LO. At N³LO, there are altogether 36 relevant parameters. Using large- N_c relations for the N³LO parameters, the number of parameters was reduced significantly down to 20. A parameter reduction by about a factor of two was achieved. Still, 20 is a large number and, in this work, we will consider a subset of the most important operators only. In this work, we will ignore the role of the five symmetry-conserving N³LO parameters. This leaves us with the 15 parameters, which we adjust to the physical baryon masses and the quark-mass dependence for the nucleon and omega mass as predicted by the BMW Collaboration [5]. With even more accurate and complete QCD lattice results it should be possible to determine the full set of large- N_c correlated parameters in the near future. The results from the BMW Collaboration are shown in Fig. 2 for three different lattice spacings. The approximate independence of the lattice spacing we take as a justification to adjust our parameters without any further continuum limit extrapolations.

Given a set of parameters there is no guarantee for a unique solution of Eq. (14) to exist. In particular, there may be a discontinuous quark-mass dependence for the baryon masses. This is a consequence of the nonlinearities in our approach as introduced by the self-consistency condition. Indeed, various discontinuities in the quark-mass dependence of the baryon masses were reported in Ref. [20] based on the chiral Lagrangian relevant at NNLO. While at NNLO, it is not possible to avoid such a discontinuous quark-mass dependence in our approach; we find that at N³LO, there are parameter sets that do lead to a smooth quark-mass dependence of the baryon masses. We observe a necessary condition for a smooth extrapolation:

$$\left. \frac{\partial}{\partial p} \Sigma_B(p) \right|_{p=M_B} < 1, \quad (21)$$

to hold for all octet and decuplet self-energies. Owing to our self-consistency constraint, the condition (21) depends on the physical baryon masses and the parameters $\bar{\zeta}_{0,D,F}, \bar{\xi}_{0,D}$ only. In order to analyze the condition (21) in more depths, we consider the three mass

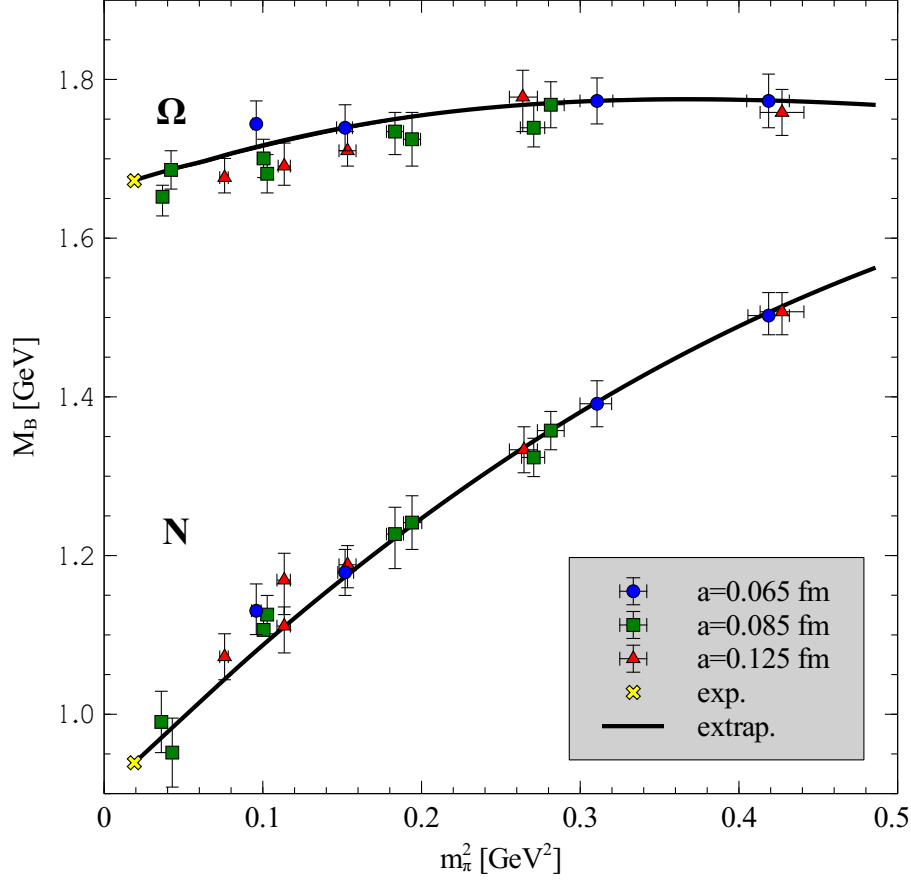


FIG. 1: Chiral extrapolation of the nucleon and omega masses. Lattice data are taken from Ref. [5].

combinations,

$$\begin{aligned}
\Delta_1 &= \frac{3}{4} M_\Lambda + \frac{1}{4} M_\Sigma - \frac{1}{2} (M_N - M_\Xi) - \frac{1}{4} (M_{\Sigma^*} - M_\Delta - M_\Omega + M_{\Xi^*}) , \\
\Delta_2 &= M_\Omega - M_{\Xi^*} - 2 (M_{\Xi^*} - M_{\Sigma^*}) + M_{\Sigma^*} - M_\Delta , \\
\Delta_3 &= M_{\Sigma^*} - M_\Sigma - M_{\Xi^*} + M_\Xi ,
\end{aligned} \tag{22}$$

studied before in Refs. [26, 28]. As shown in Ref. [26], a strict large- N_c expansion of the baryon masses at NNLO predicts $\Delta_1 = \Delta_2 = \Delta_3 = 0$. The merit of Δ_1 and Δ_2 lies in their independence of all parameters but $\bar{\zeta}_{0,D,F}$, $\bar{\xi}_{0,D}$, and $\bar{M}_{[8]}$. The mass combination Δ_3 has only an additional dependence on $\bar{d}_D - 3(\bar{b}_F + \bar{b}_D)$. These properties are a consequence of the self-consistency constraint and the large- N_c sum rules (10), which we use for the renormalized coupling constants at the renormalization scale $\mu_{UV} = \bar{M}_{[8]}$.

Using the empirical values for $\Delta_1 \simeq 3.2$ MeV and $\Delta_2 \simeq -6.1$ MeV, together with the

	Fit 1	Fit 2	Fit 3	Fit 4*	Fit 5	Fit 6
$\bar{M}_{[8]}$ [GeV]	0.9138	0.9178	0.9203	0.9111	0.9159	0.9186
$\bar{M}_{[10]}$ [GeV]	1.0937	1.0915	1.0896	1.0938	1.0917	1.0897
\bar{b}_0 [GeV ⁻¹]	-0.9115	-0.9234	-0.9355	-0.9086	-0.9201	-0.9321
\bar{b}_D [GeV ⁻¹]	0.5729	0.5978	0.6205	0.5674	0.5919	0.6141
\bar{b}_F [GeV ⁻¹]	-0.6322	-0.6542	-0.6733	-0.5880	-0.6100	-0.6289
\bar{d}_0 [GeV ⁻¹]	-0.2246	-0.2337	-0.2432	-0.2300	-0.2385	-0.2478
\bar{d}_D [GeV ⁻¹]	-0.3778	-0.3691	-0.3586	-0.3617	-0.3541	-0.3446
\bar{c}_0 [GeV ⁻³]	0.0177	0.0022	-0.0135	0.0176	0.0020	-0.0136
\bar{c}_4 [GeV ⁻³]	-0.1035	-0.2269	-0.3413	-0.1659	-0.2861	-0.3975
\bar{c}_5 [GeV ⁻³]	-0.3683	-0.3173	-0.2776	-0.3320	-0.2824	-0.2444
\bar{c}_6 [GeV ⁻³]	-1.5436	-1.5455	-1.5496	-1.2366	-1.2417	-1.2482
\bar{e}_4 [GeV ⁻³]	-0.2783	-0.2847	-0.2890	-0.2520	-0.2605	-0.2662
$\bar{\zeta}_0$ [GeV ⁻²]	1.7373	1.9201	2.0962	1.1279	1.3120	1.4885
$\bar{\zeta}_D$ [GeV ⁻²]	0.2582	0.2408	0.2244	0.2848	0.2670	0.2505
$\bar{\zeta}_F$ [GeV ⁻²]	-0.1949	-0.2221	-0.2504	-0.2221	-0.2487	-0.2769
$\bar{\xi}_0$ [GeV ⁻²]	1.1999	1.4051	1.6083	1.1964	1.4026	1.6061

TABLE III: The parameters are adjusted to reproduce the empirical values of the baryon octet and decuplet masses and the lattice results for quark-mass dependence of the nucleon and omega as shown in Fig. 1. The different parameter sets follow with $\Delta\bar{d}_D = -0.2$ GeV⁻¹ and $\Delta\bar{d}_D = -0.3$ GeV⁻¹ for the first three and last three fits, respectively. The parameter $\Delta\bar{\xi}_0$ takes the increasing values 0.2, 0.4 and 0.6 in both cases.

large- N_c sum rule $\bar{\xi}_D = 3(\bar{\zeta}_F + \bar{\zeta}_D)$ from Eq. (9), we can determine the parameters $\bar{\zeta}_{F,D}$ in terms of the two singlet parameters $\bar{\zeta}_0$ and $\bar{\xi}_0$. In turn, the condition (21) may be considered as a function of those parameters $\bar{\zeta}_0$ and $\bar{\xi}_0$ only. We find

$$\bar{\xi}_0 > \bar{\xi}_0^{\text{crit}} \simeq 1.1070 + 1.1882 \ln \bar{M}_{[8]} \quad \text{for} \quad -1.5 < \bar{\zeta}_0 < 1.5, \quad (23)$$

with $\bar{M}_{[8]}$ measured in units of GeV. The third mass combination with its empirical value

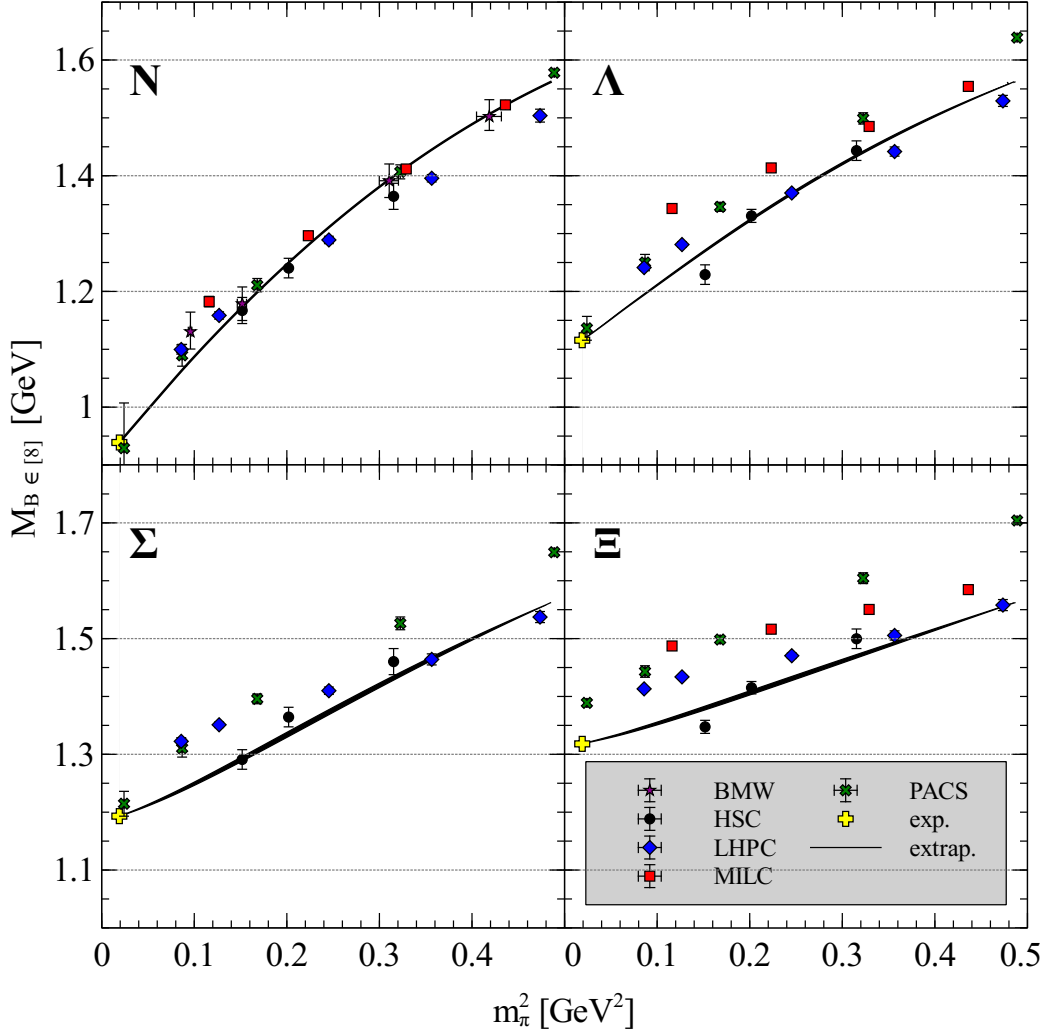


FIG. 2: Pion-mass extrapolation of the baryon octet masses.

$\Delta_3 \simeq -23.9$ MeV leads to the condition

$$\begin{aligned}
\bar{\zeta}_0 &= 3.1398 + 2.5604 \ln \bar{M}_{[8]} + 0.5800 \Delta \bar{\xi}_0 + \Delta d_D \left(6.2490 + 2.3801 \ln \bar{M}_{[8]} \right) \\
&\quad - \left(\bar{M}_{[10]} - \bar{M}_{[8]} \right) \left(0.9017 + 1.4407 \ln \bar{M}_{[8]} - 1.6120 \Delta \bar{\xi}_0 \right), \\
\bar{\xi}_0 &= \bar{\xi}_0^{\text{crit}} + \Delta \bar{\xi}_0, \quad \Delta \bar{d}_D = \bar{d}_D - 3 \bar{b}_F - 3 \bar{b}_D,
\end{aligned} \tag{24}$$

where all parameters are assumed in units of GeV. From the large- N_c sum rule (7) we expect $\Delta \bar{d}_D = 0$. As seen from (24) this would lead to unnaturally large values for $\bar{\zeta}_0$, at least for reasonable choices of $\bar{M}_{[8]}$ and $\bar{M}_{[10]}$. Since the parameter \bar{d}_D enters at NLO it is justified to admit a small $\Delta \bar{d}_D < 0$, as it would arise at the next order in the large- N_c expansion.

In the following, we assume fixed values for $\Delta \bar{d}_D$ and $\Delta \bar{\xi}_0$ and adjust the remaining 14

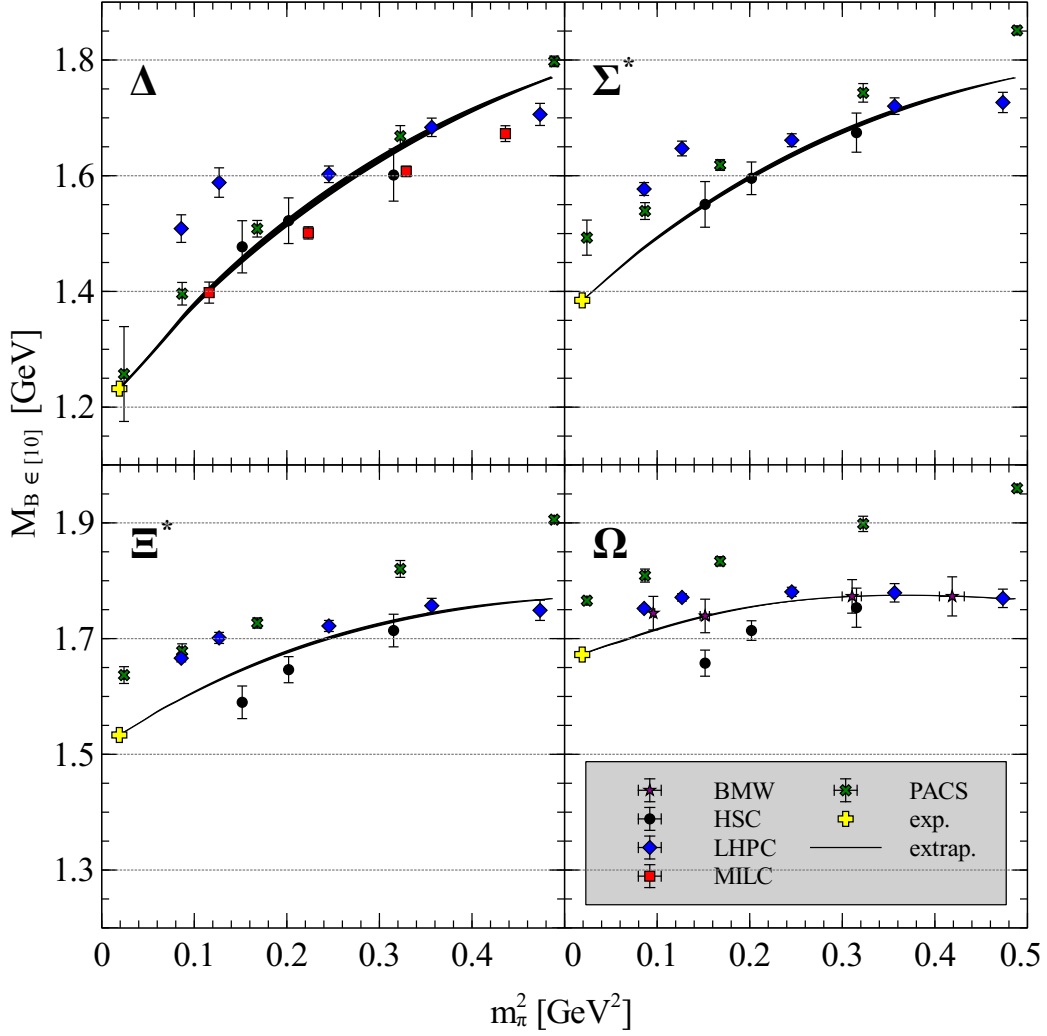


FIG. 3: Pion-mass extrapolation of the baryon decuplet masses.

parameters to the physical baryon masses and the pion-mass dependence of the nucleon and omega masses as predicted by the BMW Collaboration. The results are shown in Fig. 1 for various choices. We find that the pion-mass dependence of the BMW results can be reproduced accurately for any given $\Delta\bar{d}_D$ and $\Delta\bar{\xi}_0$. The size of the fitted parameters are collected in Table III. While the parameters $\Delta\bar{d}_D$ and $\Delta\bar{\xi}_0$ cannot be determined from the BMW results, the request for natural-size parameters favors Fit 4, for which all parameters take a reasonable size. The chiral limit values of the baryon octet and decuplet states, M and $M + \Delta$, follow from the solution of the set of nonlinear equations (20). Using the parameters of Table III, we find the ranges

$$M = 943.9 \pm 1.8 \text{ MeV}, \quad M + \Delta = 1085.8 \pm 1.6 \text{ MeV}. \quad (25)$$

In Fig. 2 and 3 we confront our results for the baryon octet and decuplet masses with the predictions of various lattice groups. The almost invisible bands in the figure are generated by the 6 parameter sets as specified in Table III. We refrain from incorporating any finite lattice effects in our present study, so the comparison in Figs. 2 and 3, is in part, of a qualitative nature. The spread in the various lattice simulation results may be taken as an indication on the size of different finite lattice effects. Most interesting is the comparison of our results with the predictions from HSC [4], for which one may expect the need of only minor lattice corrections. We find it encouraging that our approach appears to recover the pion-mass dependence of the unfitted baryon masses of HSC [4] reasonably well.

V. SUMMARY

We have studied the pion-mass dependence of the baryon octet and decuplet masses based on the chiral Lagrangian truncated at $N^3\text{LO}$. The large number of parameters was reduced significantly in application of large- N_c sum rules and therewith allowed for a first meaningful analysis of recent QCD lattice results. Altogether, we considered 16 parameters, where we ignored the small effects of the 5 symmetry-conserving $N^3\text{LO}$ parameters. In our analysis, we used a covariant form of the chiral Lagrangian and the pertinent loop functions relevant at $N^3\text{LO}$. Owing to a self-consistency condition, which requires the use of physical masses in the one-loop functions, a successful reproduction of the recent results of the BMW Collaboration on the nucleon and omega mass was achieved. A smooth quark-mass dependence arose upon a suitable choice of the symmetry-breaking $N^3\text{LO}$ parameters. A prediction for the pion-mass dependence of the remaining octet and decuplet masses was presented and confronted with available unquenched three-flavor simulations of various lattice groups. We recover the recent results of the HSC without any further adjustments.

With additional lattice data, in particular, on the dependence of the baryon masses on the strange quark mass, it should be possible to determine the remaining five symmetry-conserving parameters and scrutinize in more depths the reliability of the assumed large- N_c sum rules.

Appendix A: Tree-level baryon self-energy

We specify the renormalized tree-level self-energies for the baryon octet and decuplet states. There are several contributions to $\Sigma^{\text{tree-level}}$ in our scheme. We express the tree-level self-energy in terms of the renormalized coupling constants. It holds:

$$\begin{aligned}
\Sigma_N^{\text{tree-level}} &= -4 B_0 \left(\bar{b}_0^{\text{eff}} (2m + m_s) + \bar{b}_D^{\text{eff}} (m + m_s) + \bar{b}_F^{\text{eff}} (m - m_s) \right) \\
&\quad -4 B_0^2 \left(\bar{c}_0 (2m^2 + m_s^2) + \bar{c}_2 (m^2 + m_s^2) + \bar{c}_3 (m^2 - m_s^2) \right) \\
&\quad -2 B_0 \left(\bar{\zeta}_0 (2m + m_s) + \bar{\zeta}_D (m + m_s) + \bar{\zeta}_F (m - m_s) \right) (M_N - \bar{M}_{[8]}), \\
\Sigma_\Lambda^{\text{tree-level}} &= -4 B_0 \left(\bar{b}_0^{\text{eff}} (2m + m_s) + \frac{2}{3} \bar{b}_D^{\text{eff}} (m + 2m_s) \right) \\
&\quad -4 B_0^2 \left(\bar{c}_0 (2m^2 + m_s^2) + \frac{2}{3} \bar{c}_1 (m - m_s)^2 + \frac{2}{3} \bar{c}_2 (m^2 + 2m_s^2) \right) \\
&\quad -2 B_0 \left(\bar{\zeta}_0 (2m + m_s) + \frac{2}{3} \bar{\zeta}_D (m + 2m_s) \right) (M_\Lambda - \bar{M}_{[8]}), \\
\Sigma_\Sigma^{\text{tree-level}} &= -4 B_0 \left(\bar{b}_0^{\text{eff}} (2m + m_s) + 2 \bar{b}_D^{\text{eff}} m \right) \\
&\quad -4 B_0^2 \left(\bar{c}_0 (2m^2 + m_s^2) + 2 \bar{c}_2 m^2 \right) \\
&\quad -2 B_0 \left(\bar{\zeta}_0 (2m + m_s) + 2 \bar{\zeta}_D m \right) (M_\Sigma - \bar{M}_{[8]}), \\
\Sigma_\Xi^{\text{tree-level}} &= -4 B_0 \left(\bar{b}_0^{\text{eff}} (2m + m_s) + \bar{b}_D^{\text{eff}} (m + m_s) - \bar{b}_F^{\text{eff}} (m - m_s) \right) \\
&\quad -4 B_0^2 \left(\bar{c}_0 (2m^2 + m_s^2) + \bar{c}_2 (m^2 + m_s^2) - \bar{c}_3 (m^2 - m_s^2) \right) \\
&\quad -2 B_0 \left(\bar{\zeta}_0 (2m + m_s) + \bar{\zeta}_D (m + m_s) - \bar{\zeta}_F (m - m_s) \right) (M_\Xi - \bar{M}_{[8]}), \quad (\text{A1})
\end{aligned}$$

and

$$\begin{aligned}
\Sigma_\Delta^{\text{tree-level}} &= -4 B_0 \left(\bar{d}_0^{\text{eff}} (2m + m_s) + \bar{d}_D^{\text{eff}} m \right) \\
&\quad -4 B_0^2 \left(\bar{e}_0 (2m^2 + m_s^2) + \bar{e}_2 m^2 \right) \\
&\quad -2 B_0 \left(\bar{\xi}_0 (2m + m_s) + \bar{\xi}_D m \right) (M_\Delta - \bar{M}_{[10]}), \\
\Sigma_{\Sigma^*}^{\text{tree-level}} &= -4 B_0 \left(\bar{d}_0^{\text{eff}} (2m + m_s) + \frac{1}{3} \bar{d}_D^{\text{eff}} (2m + m_s) \right) \\
&\quad -4 B_0^2 \left(\bar{e}_0 (2m^2 + m_s^2) + \frac{1}{3} \bar{e}_1 (m - m_s)^2 + \frac{1}{3} \bar{e}_2 (2m^2 + m_s^2) \right) \\
&\quad -2 B_0 \left(\bar{\xi}_0 (2m + m_s) + \frac{1}{3} \bar{\xi}_D (2m + m_s) \right) (M_{\Sigma^*} - \bar{M}_{[10]}), \\
\Sigma_{\Xi^*}^{\text{tree-level}} &= -4 B_0 \left(\bar{d}_0^{\text{eff}} (2m + m_s) + \frac{1}{3} \bar{d}_D^{\text{eff}} (m + 2m_s) \right) \\
&\quad -4 B_0^2 \left(\bar{e}_0 (2m^2 + m_s^2) + \frac{1}{3} \bar{e}_1 (m - m_s)^2 + \frac{1}{3} \bar{e}_2 (m^2 + 2m_s^2) \right),
\end{aligned}$$

$$\begin{aligned}
& -2 B_0 \left(\bar{\xi}_0 (2m + m_s) + \frac{1}{3} \bar{\xi}_D (m + 2m_s) \right) \left(M_{\Xi^*} - \bar{M}_{[10]} \right), \\
\Sigma_{\Omega}^{\text{tree-level}} = & -4 B_0 \left(\bar{d}_0^{\text{eff}} (2m + m_s) + \bar{d}_D^{\text{eff}} m_s \right) \\
& -4 B_0 \left(\bar{e}_0 (2m^2 + m_s^2) + \bar{e}_2 m_s^2 \right) \\
& -2 B_0 \left(\bar{\xi}_0 (2m + m_s) + \bar{\xi}_D m_s \right) \left(M_{\Omega} - \bar{M}_{[10]} \right), \tag{A2}
\end{aligned}$$

with

$$\begin{aligned}
\bar{b}_0^{\text{eff}} & \equiv \bar{b}_0 + \bar{c}_6 B_0 (2m + m_s), & \bar{b}_D^{\text{eff}} & \equiv \bar{b}_D + \bar{c}_4 B_0 (2m + m_s), \\
\bar{b}_F^{\text{eff}} & \equiv \bar{b}_F + \bar{c}_5 B_0 (2m + m_s), \\
\bar{d}_0^{\text{eff}} & \equiv \bar{d}_0 + \bar{e}_4 B_0 (2m + m_s), & \bar{d}_D^{\text{eff}} & \equiv \bar{d}_D + \bar{e}_3 B_0 (2m + m_s). \tag{A3}
\end{aligned}$$

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